Tax Leakage and the Rule of t Over 1-t

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Introduction

Many organizations that forecast tax revenue use "static analysis." That is, they assume that, when tax rates change, this does not affect the income or other revenue that is being taxed. Accordingly, such analysis assumes that, e.g., when tax rates increase, people do *not* work less in response.

The product rule from calculus, however, illustrates how questionable the assumption is. To see this, consider the following simple model. Let t be the (flat) tax rate, and let $Y(t)$ be national income. Thus, total tax revenue is $tY(t)$. If the tax rate increases by a small amount, Δt, then the increase in tax revenue is, according to the product rule, approximately

$$\Delta t \cdot Y(t) + \Delta t \cdot t \cdot Y'(t).$$

Static analysis assumes that the increase is $\Delta t \cdot Y(t)$—that is, that the second term in the above expression does not exist.

In this paper I examine the leakage of tax increases. By *leakage* I mean the amount by which static analysis misestimates the increase in tax revenue from a rate increase. More specific, it equals the amount predicted by static analysis (the first term in the above expression) minus the true amount (the entire expression). Thus, it equals the negative of the second term in the expression; i.e. $-\Delta t \cdot t \cdot Y'(t)$. I define *relative leakage* as the leakage divided by the amount that static analysis predicts. That is

$$\text{Relative Leakage} = -\frac{\Delta t \cdot t \cdot Y'(t)}{\Delta t \cdot Y(t)} = -t\frac{Y'(t)}{Y(t)}.$$

I show that, under a broad set of assumptions, the latter term equals

$$\frac{t}{1-t}\frac{1-t_L}{t_L},$$

where t_L is the *Laffer rate*; i.e. the tax rate that maximizes government revenue.

The formula gives a simple method to estimate relative leakage from a tax increase. For instance, suppose the highest marginal rate for U.S. taxpayers is 36%, and suppose Congress is considering increasing the rate by a small amount. Suppose that the Congressional Budget Office,

Table 1: Relative Leakage as a function of the Laffer Rate

Laffer Rate	Relative Leakage
40%	84.4%
50%	56.3%
60%:	37.5%
70%	24.1%
80%	14.1%
90%	6.3%

Key: The table illustrates the relative leakage of a tax increase as a function of the Laffer rate. The numbers in the table assume that the current rate is 36% and that the government increases this rate by a small amount. The relative leakage is the amount by which static analysis overestimates tax revenue. For instance, if the Laffer rate is 60%, then 37.5% of the static-analysis estimate will leak. That is, one obtains a more accurate estimate of the revenue by decreasing the static-analysis estimate by 37.5%.

using static analysis, estimates that the increase will raise an extra $100 billion. My result allows one to calculate by how much the estimate will be inaccurate.

To calculate this, one only needs to know the Laffer rate. As an example, suppose it is 70%. Then, as my formula implies, the relative leakage is

$$\frac{.36}{1-.36}\frac{1-.7}{.7} = .241.$$

That is, the CBO estimate will be off by 24.1%, and thus, a more accurate forecast is that the tax increase will generate additional revenue of only $75.9 billion.

Table 1 shows other estimates of the relative leakage when instead the Laffer rate is various other rates.

All the estimates are greater than zero—thus suggesting that static analysis is not a proper method for estimating tax revenue. Meanwhile, conservative commentators sometimes claim that tax decreases will pay for themselves. The fact that all the numbers in the table are less than 100% casts doubt on such claims.

Intuition of the Main Result

My main result is that the relative leakage is proportional to $t/(1-t)$. Thus, the leakage involves two aspects: (i) it varies directly with t, and (ii) it varies inversely with $1-t$. I call aspect (i) the Willie Sutton effect, the famous bank robber. When asked why he robs banks, Sutton purportedly replied, ”Because that’s where the money is.”

One reason people may laugh at Sutton’s answer is because it ignores a simple fact: If a business has much money, then its owners will likely invest much in security to prevent a robbery.

But suppose the latter fact were not true. Suppose instead that all businesses invested the same amount in security. If so, then Sutton would be correct; the best businesses to rob would be the ones with the most money.

To see how this illustrates my result, suppose, as an example, that total income in each of the 50 U.S. states decreases by $ 1 billion. For the moment, let us ignore why total income decreases.

Instead, let us suppose that the cause is exogenous and not related to taxes. If so, by how much does each state's tax revenue fall? First, if a state's tax rate is zero percent, then revenue does not fall at all. If the rate is 10% then revenue falls by \$100 million, and if the rate is 20%, then revenue falls by \$200 million. More generally, revenue falls by $t \cdot \$1$ billion, where t is the tax rate in a state. Most important, the drop in revenue is proportional to t. That is, like Willie Sutton's banks, the greatest drop in tax revenue occurs where the tax rate is largest.

The second part of my result—aspect (ii)—is that the relative leakage varies inversely with $1-t$. To see the intuition for this, suppose, as an example, that an individual earns \$100,000 per year, and suppose that he faces a flat tax rate of 40%. Then he pays \$40,000 in taxes, and his take-home pay is \$60,000. Now suppose the government raises the tax rate by 3 percentage points to 43%. Then the individual's new take home pay is \$57,000, a decrease of 5%.

Now suppose instead that the individual faces a 90% tax rate and the government raises the rate to 93%—again, an increase of 3 percentage points. In this case his take-home pay decreases from \$10,000 to \$7,000, a decrease of 30%.

Note that the increase in taxes—3 percentage points in both cases—is more burdensome in the second case since it decreases take-home pay by a greater percentage. Meanwhile, note that take-home pay is proportional to $1-t$. Since the burden is greater when take-home pay is smaller, this means the burden varies inversely with take-home pay. Hence, the burden varies inversely with $1-t$.

1 Assumptions, Definitions, and Preliminary Results

A government charges a flat tax rate t. Each citizen $i \in I = \{1,2,\ldots,n\}$ has an exogenous wage $w_i > 0$ and chooses to work y_i fraction of a year. Thus, i's income, after paying taxes, is $w_i(1-t)y_i$. She receives utility from her income and the time she spends on leisure, $1-y_i$.

Assume her utility function, $U_i(y_i,t,w_i)$ is separable in income and leisure. This implies that there exist two functions—call them $u_i()$ and $v_i()$—such that $U_i(y_i,t,w_i) = u_i([1-t]w_iy_i) + v_i(1-y_i)$. Without loss of generality we can define utility in terms of units of leisure. This means, without loss of generality,

$$U_i(y_i,t,w_i) = u_i([1-t]w_iy_i) + 1 - y_i.$$

I assume that $u_i()$ takes a particular parametric form—specifically, a power utility function, viz., $u_i(z) = \frac{\alpha_i}{\beta} z^{\beta}$. Hence, we write i's utility as

$$U_i(y_i,t,w_i) = \frac{\alpha_i}{\beta}[(1-t)w_iy_i]^{\beta} + 1 - y_i.$$

I assume that $\alpha_i > 0$—that is, that all individuals place at least some weight on their utility from income, that it is not the case that an individual is only interested in leisure.

Although I assume a particular, parameterized form for $u_i()$, in many ways the model is very general. First, note that I do not assume a single (representative) individual. Instead, my model allows the number individuals to be any finite number.

Second, I allow β to take a continuum of possible values. This allows taxpayers' utility for income to take a variety of possible shapes. This includes linear (by allowing β to equal one) and logarithmic (by allowing β to approach zero).

Third, α_i and w_i can vary across individuals. This means that taxpayers can vary in terms of the relative importance they place on income versus leisure. It also means that my results apply to any distribution of wages.

Next, I make two assumptions to rule out counterfactual behavior. As I show, these assumptions imply restrictions on $\beta, \{\alpha_i\}$, and $\{w_i\}$.

The first assumption is that, for at least one tax rate, at least one person will work an intermediate amount—i.e. a fraction of the year that does not equal zero or one. The assumption is extremely weak. Indeed, if it did not hold, it would imply a preposterous world. It would imply that all people, no matter the tax rate, either work none at all or work every single hour of every single day of the entire year.

Assumption I: There exists a tax rate $t_0 \in (0,1)$, an individual j, and a level of work for j, $\tilde{y}_j$, such that $U(\tilde{y}_j, t_0, w_j) > U(0, t_0, w_j)$ and $U(\tilde{y}_j, t_0, w_j) > U(1, t_0, w_j)$.

The assumption implies the following restriction on β.

Lemma 1: $\beta < 1$.

Proof: Let $t_0 \in (0,1)$ and $j \in I$ be a tax rate and individual that make Assumption I true. Then for some $\tilde{y}_j \in (0,1)$,

$$U(\tilde{y}_j, t_0, w_j) > U(0, t_0, w_j) \;\&\; U(\tilde{y}_j, t_0, w_j) > U(1, t_0, w_j). \tag{1}$$

The derivative of U_j with respect to y_j is

$$\frac{\partial U_j}{\partial y_j} = \alpha_j [w_j(1-t_0)]^{\beta} y_j^{\beta-1} - 1.$$

Note that the derivative is continuous in y_j for all $y_j \in (0,1)$. This fact and (1) imply that there exists a $y^L \in (0, y_j)$ such that the derivative is positive and a $y^H \in (\tilde{y}_j, 1)$ such that the derivative is negative. This implies

$$\alpha_j [w_j(1-t_0)]^{\beta} (y^L)^{\beta-1} - 1 > 0 > \alpha_j [w_j(1-t_0)]^{\beta} (y^H)^{\beta-1} - 1.$$

Since α_j, w_j, and $1-t_0$ are positive, the above implies

$$(y^L)^{\beta-1} > (y^H)^{\beta-1},$$

which implies $\beta < 1$. ■

As the next states formally, the second derivative of $U_i()$ is negative for all $y_i \in (0,1)$.

Lemma 2: For all $t \in [0,1)$ and for all $y_i \in (0,1]$, $\partial^2 U_i / \partial y_i^2 < 0$. Further $lim_{y_i \to 0} \partial U_i / \partial y_i = \infty$.

Proof: Note that

$$\frac{\partial U_i}{\partial y_i} = \alpha_i [w_i(1-t)]^{\beta} y_i^{\beta-1} - 1.$$

By Lemma 1, $\beta - 1$ is negative. Hence, as y_i approaches zero, $y_i^{\beta-1}$ approaches positive infinity. By the assumptions of the lemma, $(1-t)$ and y_i are positive. By the assumptions of our model, α_i and w_i are positive. These facts and the above equation imply $lim_{y_i \to 0} \partial U_i / \partial y_i = \infty$. Next, note that

$$\frac{\partial^2 U_i}{\partial y_i^2} = (\beta - 1)\alpha_i [w_i(1-t)]^{\beta} y_i^{\beta-2}.$$

By the assumptions of the lemma, $(1-t)$ and y_i are positive. By the assumptions of our model, α_i and w_i are positive. By Lemma 1, $\beta - 1 < 0$. Hence $\frac{\partial^2 U_i}{\partial y_i^2} < 0$. ■

It follows from Lemma 2 that the value y_i that maximizes U_i is unique. Accordingly, we can define that value, $y_i^*(t)$, as a function of t.

Next, I assume that, no matter the tax rate, no individual spends the entire year working. The assumption is very weak. If it were not true, it would mean that at least one person is willing to work every day of the year for 24 hours each day and never sleep. The following states this assumption formally.

Assumption II: $\forall t \in [0,1]$, $\forall i \in I$, $y_i^*(t) < 1$.

As the next lemma shows, Assumption 2 implies that $\beta \geq 0$.

Lemma 3: $\beta \geq 0$.

Proof: To prove the lemma, assume to the contrary that β is negative. Let i be an arbitrary individual, and define

$$\bar{t} = max\{0, 1 - \Big(\frac{1}{\alpha_i w_i^{\beta}}\Big)^{1/\beta}\}.$$

It follows that $\bar{t} \in [0,1)$. Importantly, $\bar{t}$ is strictly less than one. I show that as long as the tax rate is greater than $\bar{t}$, i will want to set $y_i = 1$, thus violating Assumption II.

To do this, let $\tilde{t}$ be a tax rate in $(\bar{t}, 1]$. From the definitions of $\tilde{t}$ and $\bar{t}$,

$$\tilde{t} > \bar{t} > 1 - \Big(\frac{1}{\alpha_i w_i^{\beta}}\Big)^{1/\beta},$$

which implies

$$1 - \tilde{t} < \left(\frac{1}{\alpha_i w_i^\beta} \right)^{1/\beta}.$$

Since β is negative, the above implies

$$(1 - \tilde{t})^\beta > \frac{1}{\alpha_i w_i^\beta},$$

which implies

$$\alpha_i w_i^\beta (1 - \tilde{t})^\beta > 1. \tag{2}$$

Note that when the tax rate is $\tilde{t}$, the derivative of $U_i()$ with respect to y_i is

$$\frac{\partial U_i}{\partial y_i} = \alpha_i w_i^\beta (1 - \tilde{t})^\beta y_i^{\beta - 1} - 1.$$

Since β is negative, the above function is decreasing in y_i. Hence, it is minimized when $y_i = 1$. Hence

$$\frac{\partial U_i}{\partial y_i} \geq \left. \frac{\partial U_i}{\partial y_i} \right|_{y_i = 1} = \alpha_i w_i^\beta (1 - \tilde{t})^\beta - 1.$$

The above and (2) imply that, when the tax rate is $\tilde{t}$ and $y_i \in (0, 1]$,

$$\frac{\partial U_i}{\partial y_i} > 0.$$

Further, as shown by Lemma 2, as y_i approaches 0, the derivative approaches positive infinity. Consequently, $U_i()$ is maximized when $y_i = 1$.

All that we assumed about $\tilde{t}$ is that it is an arbitrary tax rate in $(\bar{t}, 1]$. Hence, we've shown that, for any tax rate above $\bar{t}$, individual i will want to work the entire year, a contradiction of Assumption II. It follows that $\beta \geq 0$. ■

As the next lemma shows, Assumption II places restrictions on α_i and w_i.

Lemma 4: For all $i \in I$, $\alpha_i w_i^\beta < 1$.

Proof: Since Assumption II is true for all $t \in [0, 1]$, it is true when $t = 0$. Substituting $t = 0$ into $\partial U_i / \partial y_i$ gives

$$\left. \frac{\partial U_i}{\partial y_i} \right|_{t = 0} = \alpha_i w_i^\beta y_i^{\beta - 1} - 1.$$

It follows that the above expression is negative when $y_i = 1$. (To see why, suppose not—that is, that the above expression is at least zero when $y_i = 1$. By Lemma 2, the second derivative of $U_i()$ is negative for all y_i. These two facts mean that for all

$y_i \in [0,1)$, $\partial U_i/\partial y_i$ is positive. This means that i's utility is maximized when $y_i = 1$, which contradicts Assumption II.) Hence,

$$\alpha_i w_i^{\beta} - 1 < 0,$$

which proves the result. ■

Lemmas 1 and 3 imply that $\beta \in [0,1]$. Recall that $U_i()$ contains a term $1/\beta$. Thus, $U_i()$ is ill-defined when $\beta = 0$, since it means we have to divide by zero. Accordingly, I restrict β so that this will never be the case.[1]

Assumption III: $\beta \neq 0$.

The above assumption, along with Lemmas 1 and 3, imply:

Lemma 5: $\beta \in (0,1)$.

As I discuss earlier, a feature of $U_i()$ is that it includes some important, special-case parameterizations of utility. One is linear utility, which requires $\beta = 1$. My assumptions disallow this. However, my model still allows β to be arbitrarily close to 1, which means $U_i()$ can become arbitrarily close to approximating linear utility. Similarly, my model does not literally include log utility; however, it allows β to be arbitrarily close to 0. This means that $U_i()$ can become arbitrarily close to approximating log utility.

Finally, as the next lemma shows, $y_i^*(t)$ must be in $(0,1)$ for all $i \in I$, and $y_i^*(t)$ can be written as a fairly simple function of t.

Lemma 6: For all $i \in I$ and all $t \in [0,1)$, $y_i^*(t) \in (0,1)$, and

$$y_i^*(t) = (\alpha_i w_i^{\beta})^{\frac{1}{1-\beta}} (1-t)^{\frac{\beta}{1-\beta}}.$$

Proof: First, Assumption II guarantees that $y^*(t) < 1$. Next, I show that $y^*(t)$ is strictly greater than zero. To do this, consider the derivative of $U_{i()}$ with respect to y_i:

$$\frac{\partial U_i}{\partial y_i} = \alpha_i [w_i(1-t)]^{\beta} y_i^{\beta-1} - 1.$$

[1] Alternatively, I could redefine $U_i()$ and remove the $1/\beta$ term. If so, then, when $\beta = 0$, $U_i(y_i,t,w_i) = \alpha_i w_i(1-t)y_i^0 + 1 - y_i = \alpha_i w_i(1-t) + 1 - y_i$, and hence $U_i()$ is maximized when $y_i = 0$. That is, for all tax rates, all individuals would choose not to work at all, which would violate Assumption I.

Since (by Lemma 3) $\beta < 1$, $y^{\beta-1}$ approaches positive infinity as y_i approaches zero. Hence, the derivative of $\partial U_i/\partial y_i$ is positive at $y_i = 0$, which means $y_i^*(t) > 0$.

Since (by Lemma 2), the second derivative of $U_i()$ is negative for all y_i, $U_i()$ is maximized at a unique point. Setting the first derivative of $U_i()$ equal to zero and simplifying gives

$$y_i^*(t) = (\alpha_i w_i^{\beta})^{\frac{1}{1-\beta}}(1-t)^{\frac{\beta}{1-\beta}}.$$

■

Next, define national income, $Y(t)$, as

$$Y(t) = \sum_{i=1}^{n} y_i^*(t),$$

and define $Y'(t)$ as its derivative.

As I discuss earlier, define the *relative leakage* of a tax increase as

$$-t\frac{Y'(t)}{Y(t)}.$$

Define $R(t) \equiv tY(t)$ as the total tax revenue, and define t_L as the it Laffer rate, the tax rate that maximizes $R(t)$.

As the next lemma shows, when $t = 1$, $y_i^*(t) = 0$—i.e. no one works. Hence, it follows that $t_L < 1$.

Lemma 7: For all $i \in I, y_i^*(1) = 0$. Hence, $t_L < 1$.

Proof: Note that when $t = 1$,

$$U_i(y_i, t, w_i) = 1 - y_i.$$

Hence, $U_{i()}$ is maximized when $y_i = 0$. Hence, $y_i^*(1) = 0$. Hence, $R(1) = 0$. By Lemma 6, for all $t \in (0,1)$, $y_i^*(t)$ is positive. Hence, R(t) is positive for all $t \in (0,1)$. Thus, $R(t)$ is not maximized at $t = 1$. Hence, $t_L < 1$. ■

2 Main Results

As the next proposition proves, the above assumptions and definitions imply that the Laffer rate equals $1-\beta$.

Proposition 1: $t_L = 1-\beta$.

Proof: By Lemma 7, $t = 1$ does not maximize $R(t)$. Hence, when we are looking for the t that maximizes $R(t)$ we can restrict ourselves to the interval $[0,1)$. Using the definition of $R(t)$ and Lemma 6's expression for $y_i^*(t)$ gives

$$R(t) = t\sum_{i=1}^{n}(\alpha_i w_i^{\beta})^{\frac{1}{1-\beta}}(1-t)^{\frac{\beta}{1-\beta}}.$$

Thus,

$$\begin{aligned}
R'(t) &= \sum_{i=1}^{n}(\alpha_i w_i^{\beta})^{\frac{1}{1-\beta}}(1-t)^{\frac{\beta}{1-\beta}} + t\sum_{i=1}^{n}(\alpha_i w_i^{\beta})^{\frac{1}{1-\beta}}(1-t)^{\frac{\beta}{1-\beta}-1}(-1)\frac{\beta}{1-\beta} \\
&= \sum_{i=1}^{n}(\alpha_i w_i^{\beta})^{\frac{1}{1-\beta}}(1-t)^{\frac{\beta}{1-\beta}} - \frac{t}{1-t}\frac{\beta}{1-\beta}\sum_{i=1}^{n}(\alpha_i w_i^{\beta})^{\frac{1}{1-\beta}}(1-t)^{\frac{\beta}{1-\beta}} \\
&= Y(t) - \frac{t}{1-t}\frac{\beta}{1-\beta}Y(t) \\
&= \left[1 - \frac{t}{1-t}\frac{\beta}{1-\beta}\right]Y(t).
\end{aligned}$$

By Lemma 6, $y_i^*(t)$ is positive for all $i \in I$. Hence, $Y(t)$ is positive. This means that $R'(t)$ is positive if and only if the term in brackets is positive. Some algebra shows that the latter is true if and only if $t < 1-\beta$.[2] Moreover, it follows that: (i) $R'(t)$ is positive for all $t \in [0, 1-beta)$; (ii) $R'(t) = 0$ when $t = 1-\beta$; and (iii) $R'(t)$ is negative for all $t \in (1-\beta, 1)$. The latter statement implies that $R(t)$ is maximized at $t = 1-\beta$. Hence, $t_L = 1-\beta$.

■

I can now state the main result of the paper, that the relative leakage of a tax increase equals $\frac{t}{1-t}\frac{1-t_L}{t_L}$.

Proposition 2: The relative leakage of a tax increase is $\frac{t}{1-t}\frac{1-t_L}{t_L}$.

[2] The term in brackets is positive iff $\frac{1-t}{t} > \frac{\beta}{1-\beta}$, which is true iff $\frac{1}{t} - 1 > \frac{\beta}{1-\beta}$, which is true iff $\frac{1}{t} > \frac{1-\beta}{1-\beta} + \frac{\beta}{1-\beta}$, which is true iff $\frac{1}{t} > \frac{1}{1-\beta}$, which is true iff $t < 1-\beta$.

Proof: Recall that the relative leakage of a tax increase is defined to be

$$-t\frac{Y'(t)}{Y(t)}.$$

By Lemma 6 and our definition of $Y(t)$,

$$Y(t) = \sum_{i=1}^{n} (\alpha_i w_i^{\beta})^{\frac{1}{1-\beta}} (1-t)^{\frac{\beta}{1-\beta}}.$$

Thus,

$$\begin{aligned} Y'(t) &= \sum_{i=1}^{n} (\alpha_i w_i^{\beta})^{\frac{1}{1-\beta}} (1-t)^{\frac{\beta}{1-\beta}-1} (-1) \frac{\beta}{1-\beta} \\ &= -\frac{\beta}{1-\beta}\frac{1}{1-t} Y(t). \end{aligned}$$

Hence, $Y'(t)/Y(t) = -\frac{\beta}{1-\beta}\frac{1}{1-t}$, and the relative leakage equals

$$\frac{t}{1-t}\frac{\beta}{1-\beta}.$$

By Proposition 1, $\beta = 1 - t_L$. Substituting this into the above expression proves the result. ■